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Research Article

Assessment of Argumentation and Proof-Based Mathematics Instruction

Argümantasyon ve İspat Tabanlı Matematik Öğretiminin Değerlendirilmesi

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Abstract

Purpose: *One of the fundamental goals of educational systems is to cultivate individuals who are capable of accessing information and critically evaluating its accuracy. The ability to engage in logical reasoning and to present arguments coherently and persuasively has become an increasingly essential competency in the current societal context. This study aims to examine students' argumentation schemes, proof schemes, and their perspectives within the context of Argumentation and Proof-Based Mathematics Instruction.* ***Method:*** *A qualitative research design was employed to explore the impact of Argumentation and Proof-Based Mathematics Instruction (APBMI). The study was conducted with 30 sixth-grade students in Kayseri, Türkiye. To conduct a detailed analysis of qualitative data, 12 students were selected from among the thirty students and interviewed. A 20-hour instructional program was designed and implemented over six weeks.* ***Findings:*** *Based on the analysis of students' responses, two levels of argumentation quality, "demonstrate an acceptable level of argumentation quality" and "demonstrate a good level of argumentation quality," were identified. In addition, students' proof schemes were analyzed using specially designed proof tasks. Students were able to construct external, experimental, and analytical proof schemes. Content analysis of the interview data revealed three main themes: usefulness, effectiveness, and collaborative learning.* ***Conclusion:*** *The results indicate that students demonstrated varying levels of competence in argumentation processes. Regarding APBMI, students generally held positive views concerning its usefulness, effectiveness, and capacity to support collaborative learning. The fact that students produced evidence at external, experimental, and analytical levels reflects the diversity of their proof skills. Taken together, these findings suggest that APBMI can make a positive contribution to classroom practices in mathematics instruction.*

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Öz

Amaç: *Eğitim sistemlerinin temel hedeflerinden biri bilgiye erişebilen ve doğruluğunu eleştirel bir şekilde değerlendirebilen bireyler yetiştirmektir. Mantıklı akıl yürütme ve ikna edici argümanlar sunma yeteneği, günümüz toplumlarında giderek önemi artan bir beceri haline gelmiştir. Bu çalışmanın amacı, Argümantasyon ve İspat Tabanlı Matematik Öğretimi (AİTMÖ) bağlamında öğrencilerin argümantasyon şemalarını, kanıt şemalarını ve uygulamaya yönelik düşüncelerini incelemektir.* ***Yöntem:*** *Bu araştırmada nitel bir araştırma tasarımı kullanılmıştır. Çalışma Türkiye'de Kayseri ilinin bir merkez ilçesinde öğrenime devam eden 30 altıncı sınıf öğrencisi ile gerçekleştirilmiştir. Nitel verinin derinlemesine analiz edilmesi amacıyla bu otuz öğrenci arasından 12 öğrenci belirlenmiş ve görüşmeler yapılmıştır. Altı hafta süren 20 saatlik bir öğretim tasarlanmış ve uygulanmıştır.* ***Bulgular:*** *Öğrencilerin argümantasyon kaliteleri, verilen yanıtlara göre, "kabul edilebilir düzeyde argümantasyon kalitesine sahip" veya "iyi düzeyde argümantasyon kalitesine sahip" olarak belirlenmiştir. Buna ek olarak, öğrencilerin ispat şemaları da özel olarak hazırlanmış ispat görevleri aracılığı ile incelenmiştir. Bulgular, öğrencilerin dışsal, deneysel ve analitik kanıt şemaları oluşturabildiklerini göstermiştir. Görüşme verilerine yönelik yapılan içerik analizi sonucunda belirlenen üç ana tema: kullanışlılık, etkililik ve işbirlikli öğrenme olmuştur.* ***Sonuç:*** *Sonuçlar öğrencilerin argümantasyon süreçlerinde çeşitli düzeylerde yetkinlik sergilediklerini göstermektedir. AİTMÖ'ye yönelik olarak; kullanışlık, etkililik ve işbirlikli öğrenme açısından, genel olarak olumlu bakış açılarına sahip olduklarını göstermektedir. Öğrencilerin dışsal, deneysel ve analitik düzeyde kanıtlar üretmiş olmaları kanıt oluşturma becerilerinin çeşitliliğini yansıtmaktadır. Bu bulgular birlikte ele alındığında, Argümantasyon ve İspat Temelli Matematik Öğretiminin matematik derslerinde sınıf içi uygulamalara olumlu katkısının olacağı düşünülmektedir.*

Introduction

The ability to reason logically and present arguments persuasively has become increasingly essential. Arguments are discourses that provide justification or explanation for an asserted claim (Willard, 2003), and they are developed to support or plausibly explain a particular point of view (Kuhn, 1993). Argumentation refers to the process of reasoning about a proposition, generating hypotheses, sharing intuitions, and constructing arguments based on statements and ideas presented individually or collaboratively (Güneş, 2013). In the process of argumentation, individuals are expected to substantiate the validity of their reasoning, either to themselves or to others. However, not all forms of discourse constitute argumentation in the academic or scientific sense. Scientific argumentation specifically entails the structured presentation of claims, the formulation of responses to those claims, the provision and evaluation of evidence and justifications, and the systematic analysis of the reasoning process as a whole (Nickerson, 2020).

Argumentation is not merely a means of reaching conclusions, but also a conceptual process of deriving reasoned inferences through the systematic consideration of premises that substantiate the validity of those conclusions (Toulmin, 2003). It involves not only the construction of arguments but also the cognitive skills necessary for understanding, evaluating, and engaging with them, particularly in collaborative or group-reasoning contexts (Mercier & Sperber, 2011). The argumentation process requires classroom interaction, students' ability to express their opinions, the development of critical perspectives, and the collective construction of mathematical knowledge (Vygotsky, 2012, 1986).

In the context of mathematics, argumentation facilitates a detailed and accurate understanding through structured discussion and reasoning around a given mathematical proposition. Consistent with Toulmin's (2003) framework, argumentation involves establishing the validity of a claim or proposition by assuming the truth of supporting statements. Within educational research, Toulmin's model has been widely employed to analyze learning processes and to examine the structure and progression of classroom discussions (Krummheuer, 2000). Toulmin's model consists of six elements: claim, grounds, warrant, qualifier, rebuttal, and backing. While claim, grounds, and warrant are present in every argument, qualifiers, backing, and rebuttals are present in the extended model and do not necessarily occur consistently (Rienks et al., 2005). The model has also been used in mathematics to compare the content of arguments and analyze the quality of mathematical proof (Pedemonte, 2007). Since arguments consist of statements or sentences containing claims (Shapiro & Kissel, 2022), the argumentation process is related to proof (Pedemonte, 2008).

The National Council of Teachers of Mathematics (NCTM, 2000) emphasizes the need to increase argumentation and proof-based activities in mathematics teaching by introducing argumentation and proof as the foundation of mathematics to all students from preschool to 12th grade, conducting mathematical conjectures and investigations, developing and evaluating mathematical arguments and proofs, and providing opportunities to select and use various reasoning and proof methods. The application of the argumentation process to solving a problem or developing a hypothesis supports the development of mathematical proofs (Bussi, 2009). Despite this, argumentation differs from proof because it is informal and discursive, whereas mathematics relies on formal proofs. Therefore, a mathematical proof is considered an argument approved by the mathematical community (Weber, 2014). However, not all arguments are expected to be accurate. Producing proof not only enhances students' mathematical potential but also develops the skills they will need in all situations that require inference and decision-making (Fawcett, 1938).

In mathematics education, argumentation and proof represent two fields with distinct conceptual frameworks. Mathematics provides a valuable context for presenting arguments, defending them in social interactions, and developing argumentation skills (OECD, 2018). Many studies focus on argumentation to support classroom discourse,

while others emphasize proof as a formal mathematical practice. However, there is limited research on instructional approaches that explicitly integrate argumentation and proof within the same teaching process. Therefore, this study aims to address this gap by examining the implementation of Argumentation and Proof-Based Mathematics Instruction (APBMI). The dialogical aspect of argumentation, which supports classroom discussions, plays an important role in meaningful learning by enabling students to seek information and critically question its validity (Rapanta, 2019). Moreover, constructing proofs not only enhances students' mathematical potential but also develops skills essential for situations requiring inference and decision-making (Fawcett, 1938). One of the major challenges in mathematics education is helping students develop the ability to generalize from mathematical structures (Stylianides, 2008). According to Harel and Sowder (2007), one reason students struggle is the limited emphasis on proof construction activities in school mathematics.

Literature review/Theoretical background

The production of mathematical knowledge is associated with the development of auxiliary mathematical objects such as theorems, algorithms, proofs, explanations, and justifications (Bereiter & Scardamalia, 1996), and mathematical understanding can be developed through dialogues such as collaborative discussions (Carrascal, 2015). Extensive studies have examined the impact of making proofs, such as inductive and deductive proofs in mathematics education (Balacheff, 2002; Lakatos, 1963; Polya, 1990). Proof in mathematics helps ensure the continuity between elementary and middle school mathematics and helps identify misconceptions (Stylianides & Stylianides, 2018). Harel and Sawder (2007) defined proof as mental actions undertaken to resolve or raise doubts about the truth of a claim. They used proof schemes to determine the level and production processes of mathematical proofs. Harel and Sowder (1998) identified three basic types of evidence schemes: external evidence schemes (authoritarian, symbolic, and ritual), empirical evidence schemes (example-based and perceptual), and analytic evidence schemes (transformational, axiomatic). Simple mathematical proofs can be effectively represented using Toulmin's argumentation diagram (Aberdein, 2005).

Toulmin conceptualizes an argument as a series of interconnected claims or reasons that collectively determine the strength of the content under discussion, and identifies six constituent elements of an argument (Toulmin, 1984). In Toulmin's model, the core components of an argument are claims, data, and warrants, while the auxiliary components include backing, rebuttal, and qualifiers. Van Eemeren (1995) emphasized that incorporating argumentation in education contributes to the development of democratic awareness. Therefore, it is important to design systematic learning environments in which diverse ideas can be discussed and critically appreciated.

Argumentation and proof could play an important role in achieving meaningful learning (Efendioğlu & Yanpar Yelken, 2009). Additionally, some studies reveal the positive effects of using argumentation and proof (Adal, 2023; Demirci, 2023; Dierdorp, 2021; Doruk et al., 2018; Öz, 2019; Mercan, 2015; Şengül & Tavşan, 2019; Topuz &

Günhan, 2021). Argumentation and proof activities can be effective in helping students construct knowledge accurately (Erkol et al., 2017). Similarly, Güneş (2013) showed that using the argumentation process at the beginning of the mathematical proof stage is effective in producing higher-level proof structures. Almpani et al. (2023) emphasized the importance of argumentation in the discovery stage when producing mathematical proofs and explained the common aspects of argumentation and proof.

Mathematical reasoning skills are among the core skills of new curricula and constitute an important component of mathematical competence. Various studies have shown that students have misconceptions and faulty learning in geometry (Bulut & Erkan, 2020; Çavuş Erdem & Gürbüz, 2018; Kaya, 2019). Therefore, the development of argumentation and proof skills creates opportunities for students to produce planned solution strategies actively (Fielding et al., 2015; Jones & Tzekaki, 2016; Stylianides et al., 2016). Consequently, one of the goals of education systems is to cultivate individuals who know how to access information and who can verify its accuracy. This designed instruction aims to determine the quality of argumentation schemes, proof structures, and student opinions.

Research questions

1. How is the quality of students' argumentation schemes in APBMI?
2. What are the student opinions about designed instruction?
3. How is the quality of students' proof schemes within APBMI?

Methodology

In this study, a qualitative research design was used to analyze the effects of argumentation and proof-based instruction. The research sought to explore what kinds of ideas are emphasized in Argumentation and Proof-Based Mathematics Instruction (APBMI). Additionally, the research aimed to determine the level of students' proof and argumentation schemes to clarify the effectiveness of APBMI.

Participants and design

The participants were 30 sixth graders recruited from a middle school in Kayseri, Türkiye. Participants were selected using a purposive sampling method. An achievement test was conducted, and three levels (high, medium, and low) were determined. Based on the achievement test results, twelve students were assigned to groups. Semi-structured open-ended interviews were conducted with these twelve students. Six of these twelve students were asked open-ended questions to assess the level of their proof schemes. Demographic characteristics of the interviewed students are presented in Table 1.

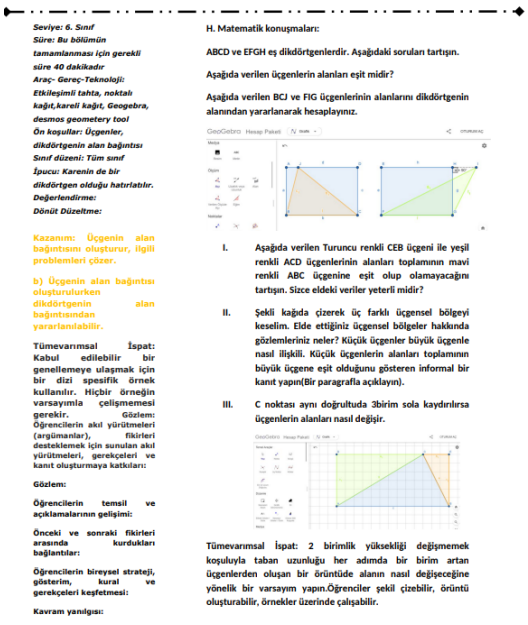
Table 1: *Demographic characteristics of the interviewed students*

Student	Grade	Achieving group
S1	6	High
S2	6	High
S3	6	High
S4	6	High
S5	6	Moderate
S6	6	Moderate
S7	6	Moderate
S8	6	Moderate
S9	6	Low
S10	6	Low
S11	6	Low
S12	6	Low

Procedure and Data Collection

Based on the document in Table 2, a 20-hour course was designed for argumentation and proof-based mathematics instruction. The activities were distributed among five groups of six students each. The teacher maintained a supportive and encouraging attitude throughout the activities. The teacher followed the prepared teacher material throughout the activity process. Instruction was carried out according to the lesson plans included in the material. An example of the prepared teacher material is presented in Figure 1.

Figure 1: *An example of teacher material*



Seviye: 6. Sınıf
Süre: Bu bölümün tamamlanması için gerekli süre 40 dakikadır
Araç- Gereç-Teknoloji:
Etkileşimli tahta, noktalı kağıt,kareli kağıt, Geogebra, desmos geometery tool
Ön koşullar: Üçgenler, dikdörtgenin alan bağıntısı
Sınıf düzeni: Tüm sınıf
İpucu: Karenin de bir dikdörtgen olduğu hatırlatılır.
Değerlendirme:
Dönüt Düzeltme:

Kazanım: Üçgenin alan bağıntısını oluşturur, ilgili problemleri çözer.

b) Üçgenin alan bağıntısı oluşturulurken dikdörtgenin alan bağıntısından yararlanılabilir.

Tümevarımsal İspat: Kabul edilebilir bir genellemeye ulaşmak için bir dizi spesifik örnek kullanılır. Hiçbir örneğin varsayımla çelişmemesi gerekir. Gözlem: Öğrencilerin akıl yürütmeleri (argümanlar), fikirleri desteklemek için sunulan akıl yürütmeleri, gerekçeleri ve kanıt oluşturmaya katkıları:

Gözlem:

Öğrencilerin temsil ve açıklamalarının gelişimi:

Önceki ve sonraki fikirleri arasında kurdukları bağlantılar:

Öğrencilerin bireysel strateji, gösterim, kural ve gerekçeleri keşfetmesi:

Kavram yanılgısı:

Gerçek Hayatla İlişkilendirme Linki:

H. Matematik konuşmaları:

ABCD ve EFGH eş dikdörtgenlerdir. Aşağıdaki soruları tartışın.

Aşağıda verilen üçgenlerin alanları eşit midir?

Aşağıda verilen BCJ ve FIG üçgenlerinin alanlarını dikdörtgenin alanından yararlanarak hesaplayınız.

I. **Aşağıda verilen Turuncu renkli CEB üçgeni ile yeşil renkli ACD üçgenlerinin alanları toplamının mavi renkli ABC üçgenine eşit olup olamayacağını tartışın. Sizce eldeki veriler yeterli midir?**

II. **Şekli kağıda çizerek üç farklı üçgensel bölgeyi keselim. Elde ettiğiniz üçgensel bölgeler hakkında gözlemleriniz neler? Küçük üçgenler büyük üçgenle nasıl ilişkili. Küçük üçgenlerin alanları toplamının büyük üçgene eşit olduğunu gösteren informal bir kanıt yapın(Bir paragrafla açıklayın).**

III. **C noktası aynı doğrultuda 3birim sola kaydırılırsa üçgenlerin alanları nasıl değişir.**

Tümevarımsal İspat: 2 birimlik yüksekliği değişmemek koşuluyla taban uzunluğu her adımda bir birim artan üçgenlerden oluşan bir örüntüde alanın nasıl değişeceğine yönelik bir varsayım yapın.Öğrenciler şekil çizebilir, örüntü oluşturabilir, örnekler üzerinde çalışabilir.

Before the application, students were informed about the argumentation and proof processes. The activities aimed to complete the argumentation schemes and produce appropriate proof structures both individually and in groups. The APBMI was completed in a classroom environment over six weeks. All tasks were distributed to groups of six students within 60 minutes. Students were divided into three groups: low, medium, and high, based on the results of an academic achievement test. This grouping enabled a differentiated analysis of learning outcomes, facilitated the connection between student performance and underlying cognitive processes, and provided insights to inform instructional design. Twelve students were selected from these three groups for qualitative analysis. Semi-structured interview questions were used to determine the quality of arguments and to gain a deeper understanding of the students' reasoning. Content analysis was applied to describe the scope of the discourses.

The quality of the arguments in the activities was evaluated through a rubric based on Toulmin's argument model (Toulmin, 1984), prepared by Cho and Jonassen (2002) and adapted into Turkish by Aydın Güç and Kuleyin (2021). The argumentation quality rubric was developed based on the criteria of claim, data, warrants, backing, qualifier, and rebuttal. Each argumentation scheme was assessed according to the extent to which it satisfied these criteria. Based on this assessment, schemes were categorized as lacking argumentation quality, exhibiting some degree of argumentation quality, demonstrating acceptable argumentation quality, or demonstrating good argumentation quality. In this study, Harel and Sowder's (1998) proof schemes were used to determine the level of mathematical proof structures. In the final phase of the application, six students were selected from these 12, aiming to achieve maximum diversity in proof schemes across the low, moderate, and high-achieving groups. The students' proofs were subsequently examined and analyzed by two experts. The implementation process of the designed instruction is presented in Table 2.

Table 2: *The process of APBMI*

Process	Activity	Method
Introduction of the Argumentation Model and Proof Methods	Toulmin's Model and introduction of mathematical proof methods (inductive, deductive)	Presentation, group work
Activity1	Draw heights of different geometric shapes and explain why they have heights.	Argumentation scheme, class discussion
Activity 2	Is it possible to calculate the area of a triangle using the area of a different geometric shape?	Argumentation scheme, group work, mathematical proofs
Activity 3	Which line segments' lengths must be known to calculate the area of △ ACD?	Group work, mathematical proofs, mathematical conversations

Activity 4	Calculate the area of the obtuse triangle given below using the area of the rectangle.	Argumentation scheme, group work, mathematical proofs
Activity 5	Discuss whether the sum of the areas of △CEB and △ACD can be equal to the △ABC. Do you think the available data is sufficient?	Group work, mathematical proofs, mathematical conversations
Activity 6	Prove that the area of a right triangle is equal to half the product of the lengths of its legs.	Group work, mathematical proofs, mathematical conversations
Activity 7	Discuss whether the side lengths and heights of two equilateral triangles with the same area must always be the same? Explain your reasons.	Group work, mathematical proofs, mathematical conversations
Activity 8	What is the relationship between the height of a parallelogram and the height of a rectangle with the same base length?	Group work, mathematical proofs, mathematical conversations
Activity 9	Can the area of a parallelogram be calculated using the area of a triangle?	Argumentation scheme, group work, mathematical proofs
Activity 10	Prove that the area of a parallelogram is the product of the length of the base and the height to that base.	Group work, mathematical proofs, mathematical conversations
Activity 11	Show the area of the parallelogram using the area of the triangle.	Group work, mathematical proofs, mathematical conversations
Activity 12	Explain how to calculate the area of the given compound shapes.	Group work, mathematical proofs, mathematical conversations

Reliability and Validity

The Cohen's Kappa coefficient of agreement for the scores given by two experts was 0.911 on argument quality. Cohen's Kappa indicates strong agreement between experts on argument quality. To determine the level of proof structures, two experts' opinions were obtained using Harel and Sowder's (1998) proof schemes. The levels of proof schemes were determined based on student interviews. Two experts were selected as coders for the content analysis of the interviews. Both coders were skilled at content analysis.

Role of the Researcher

In this study, the corresponding author conducted standardized open-ended interviews and identified codes and themes for the content analysis. Both authors participated in the recoding phase and reorganized the data. The corresponding author evaluated the argumentation schemes using a rubric and obtained an assessment from an expert mathematician; inter-rater agreement was then calculated between the experts. Based on

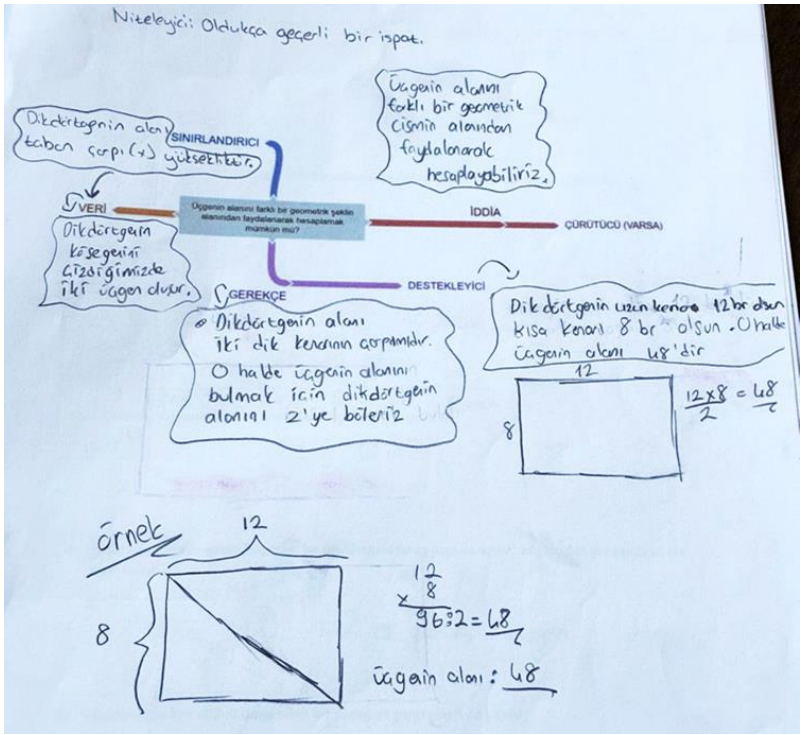
the expert's feedback, the corresponding author determined the levels of the argumentation schemes and made necessary adjustments. Finally, the corresponding author conducted interviews with the students to ensure that the argumentation schemes were clearly defined.

Results/Findings

Argumentation scheme used in the first group

The students were asked the following question: "Can the area of a triangle be found using the area of another geometric shape?" According to the document analysis, the first group presented a proposition for the claim criterion; however, it was considered incomplete. The claim is present, but it is too general. The data criterion was relevant to the claim but was also incomplete. The warrant criterion clearly explained how the data supported the claim. Regarding the backing criterion, students provided accurate, relevant, and specific sources of support. The students also presented a qualifier. However, no rebuttal element was included. Based on these findings, the students demonstrated acceptable argumentative quality. The students' argumentation elements are presented below, and the corresponding argumentation scheme is shown in Figure 2.

Figure 2: *The argumentation scheme being used in the first group*



The explanations associated with Figure 2 are presented below.

Claim: The area of a triangle can be calculated using the area of a different geometric shape.

Grounds: The area of a rectangle is calculated by multiplying the width by the Length. When we draw the diagonal of a rectangle, two triangles are formed.

Warrants: The area of a rectangle is the product of two vertical sides. Therefore, to find the area of a triangle, we divide the area of the rectangle in half.

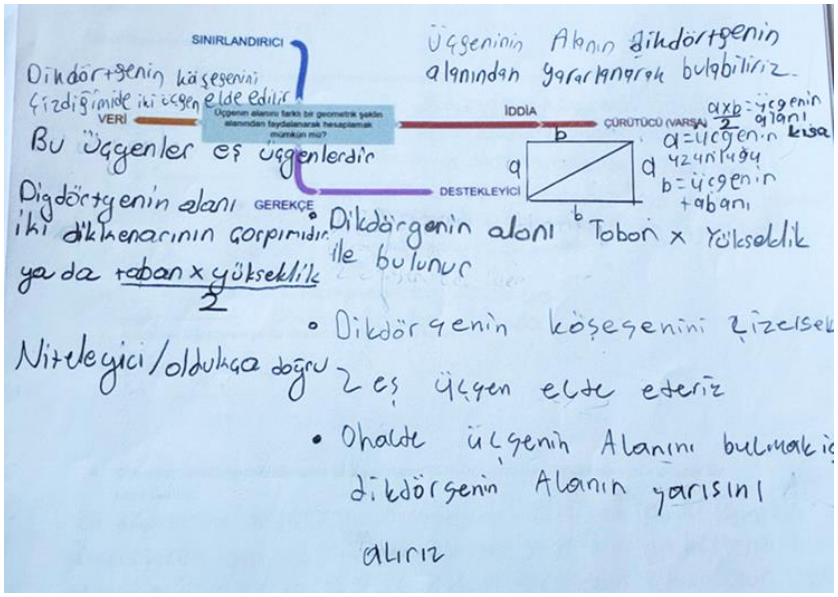
Backing: Let the longest side of the rectangle be 12 units, and the shortest side be 8 units. Therefore, the area of the triangle is 48 units.

Qualifier: A fairly valid proof.

Argumentation scheme used in the second group

The students were asked the following question: "Can the area of a triangle be found using the area of another geometric shape?" The second group's claim criterion contained clear and complete generalizations. The data criterion was relevant to the claim; however, it was not fully developed. The warrant criterion clearly explained how the data supported the claim. The backing criterion included sources of support that were accurate, relevant, and specific. However, the students did not provide a rebuttal element. The students' assigned qualifier for the proof's validity was "fairly accurate." Based on these findings, the students demonstrated acceptable argumentation quality. The students' statements are presented in the following section. The students' argumentation schemes are presented in Figure 3.

Figure 3: *The argumentation scheme being used in the second group*



The explanations for Figure 3 are presented below.

Claim: We can find the area of a triangle using the area of a rectangle.

Grounds: When the diagonal of a rectangle is drawn, two triangles are obtained. These triangles are congruent. The area of a rectangle is the product of its two perpendicular sides.

Warrants: The area of a rectangle is found by multiplying the width by the length. If we draw the diagonal of a rectangle, we obtain two congruent triangles. Therefore, to find the area of a triangle, we take half the area of the rectangle.

Backing: axb /2 (Drawing presented)

Qualifier: Quite true.

Student Opinions About APBMI

Three themes emerged from the content analysis: usefulness, effectiveness, and group work. Student perspectives and analyses related to the theme "Usefulness of APBMI," derived from the content analysis, are presented in Table 4 below. The most frequently mentioned positive code regarding the usefulness of APBMI was "creating a more engaging and enjoyable learning environment." In contrast, the most frequently mentioned negative code was "struggling to produce arguments or evidence."

Some examples of statements coded under the usefulness theme include the following: "I think the positive aspect is that it makes learning easier and more fun"; "In my opinion, learning new ideas is not difficult; it is actually enjoyable"; "This class was more fun than the others"; "It is a bit difficult to understand how each step is connected to the others"; "It is difficult to put the types of evidence we have in mind onto paper, and it is also challenging to defend and prove them"; and "It was difficult to produce a proof." Usefulness codes are presented in Table 3.

Table 3: *Usefulness theme*

Theme	Category	Codes	Students											
			1	2	3	4	5	6	7	8	9	10	11	12
Usefulness	Positive	1.Creating a more engaging and fun learning environment.	x		x	x	x	x	x	x		x		
		2.Improved clarity and comprehensibility of lessons	x			x			x					
		3.Creating solutions together		x	x	x	x	x		x	x			
		4.The student's ability to communicate ideas effectively.												x
		5.Integration into everyday situations	x	x			x	x			x			

	6.Implementation across different subject areas			x			x	x	x					
	7.Practical utility	x					x	x			x			
	8.Generating new ideas		x			x	x		x	x				
Negative	9.Struggling to provide arguments or evidence.		x		x	x	x		x	x	x	x	x	
	10.This process is time-consuming			x	x		x			x		x		
	11.Difficulty in generating ideas									x			x	
	12.Inconsistent viewpoints.	x			x	x	x		x		x	x	x	

Another theme that emerged from the interviews was "effectiveness." The most frequently mentioned positive codes regarding the effectiveness of APBMI were "ensuring better learning" and "establishing the relationship between argument and proof." In contrast, the most frequently mentioned negative code was "inability to apply argumentation to different topics."

Some examples of statements coded under the effectiveness theme include the following: "I view mathematics from a different perspective, which positively influences my attitude toward the subject because I understand the topics better"; "By discussing the questions, we gained a better understanding"; "Yes, because it helps me understand the subject better, and I can solve the questions more easily using evidence"; "I don't think I am good at coming up with arguments on different topics; I do not feel confident enough about it"; and "I cannot say that evidence always comes to mind, but maybe it could." The codes related to the effectiveness theme are presented in Table 4.

Table 4: *Efectiveness theme*

Theme	Category	Codes	Students											
			1	2	3	4	5	6	7	8	9	10	11	12
Efectiveness	Positive	1.Competence in generating proof on various subjects.		x	x	x	x	x			x			
		2.Competence in generating argument on various subjects.		x	x		x	x		x	x		x	
		3.Facilitating long-term learning	x		x		x		x					
		4.Establishing the relationship between argument and proof		x		x	x			x	x		x	x

		5.Proof as a central component of mathematics	x	x	x	x	x		x			x			
		6.Ensuring better learning		x	x	x		x	x	x	x	x			
		7.Cultivating a love for mathematics among students	x	x											
		8.Reducing mathematics-related anxiety.			x				x	x					
		9.Effective in acquiring new knowledge											x		x
		10.Relating to the lesson/reflecting	x			x	x		x		x				
	Negative	1.Inability to reflect on different subject for argumentation										x			x
		2.Inability to reflect on different subject for proof											x		

Another theme that emerged from the interviews was "collaborative learning". The most frequently mentioned positive code regarding the collaborative learning aspect of APBMI was "critically interpreting the adequacy of evidence presented by peers." In contrast, the most frequently mentioned negative codes were "failing to engage consistently and equitably in group activities" and "disagreements."

Some examples of statements coded under the collaborative learning theme include the following: "Most of the time, my groupmates' explanations matched the proof"; "Although my friends' explanations occasionally contradict one another, they are generally consistent and help guide us toward supporting our claims"; and "They were all different; some were familiar to me, while others were unfamiliar but still reasonable". The codes related to the collaborative learning theme are presented in Table 5.

Table 5: *Collaborative learning theme*

Theme	Categories	Codes	Students											
			1	2	3	4	5	6	7	8	9	10	11	12
Collaborative learning	Positive	1. Fostering cooperation and solidarity among students	x	x	x			x			x	x	x	
		2. Promoting the sharing of thoughts and perspectives.						x					x	

	3. Recognizing the thinking styles of others					x									
	4. Attempting to resolve conflicts and contradictions	x	x	x	x			x		x	x		x		
	5. Collaborative decision-making	x		x		x	x		x	x					
	6. Critically interpreting the adequacy of evidence presented by peers	x	x	x	x	x	x	x	x	x					
	7. Persuading peers by employing diverse forms of evidence.	x	x	x	x	x	x		x						
	8. Loving collaborative learning	x					x	x	x	x					
	9.Learning collaboratively			x							x		x		x
	10. Recognizing and synthesizing different viewpoints within collaborative discussions	x		x					x						x
Negative	1.Failing to engage consistently and equitably in team activities	x			x		x				x				
	2.Disagreements	x			x				x						x
	3.Competitive interactions between distinct groups.			x											

Proof schemes used in tasks

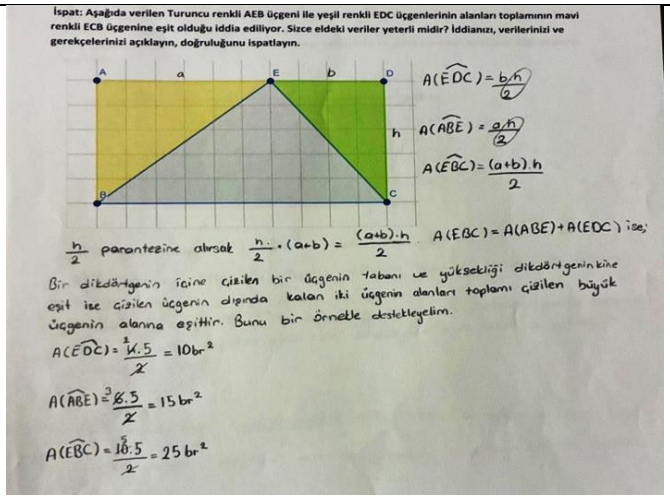
First, an academic achievement test was developed. Based on the test results, students were divided into three groups: low, medium, and high achievement levels. Twelve students were then selected from these groups for content analysis. Subsequently, two students from each achievement level were chosen to examine proof schemes, resulting in a total of six students to maximize the diversity of findings. The students were coded as S1, S2, S5, S6, S9, and S11. Among them, S1 and S3 were from the high-achievement group, S5 and S6 from the medium-achievement group, and S9 and S11 from the low-achievement group. The proof schemes of these students are presented in Table 6.

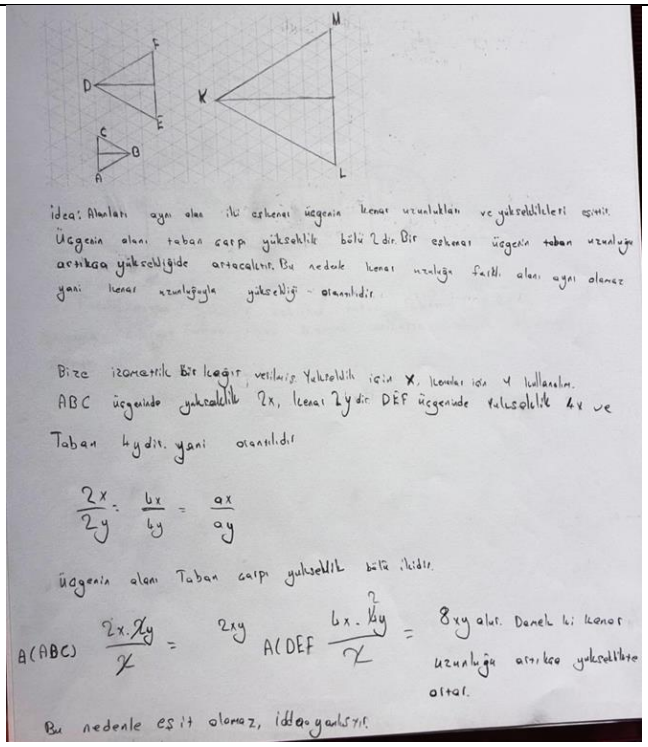
Table 6: *Students' proof schemes*

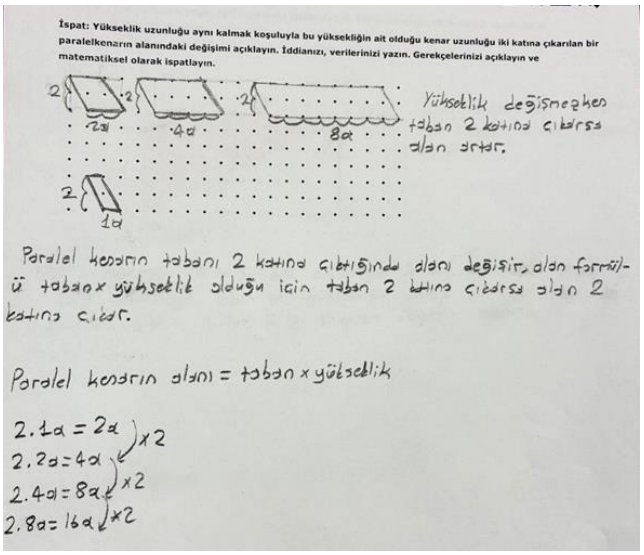
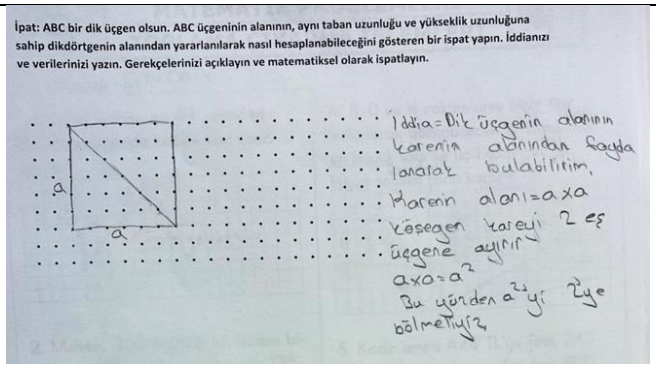
Task	S1	S3	S5	S6	S9	S11
1	transformational/ inductive	perceptual/ inductive	transformational/ inductive	perceptual	inductive/ perceptual	inductive
2	transformational/ inductive	perceptual/ inductive	inductive	perceptual	symbolic/ authoritarian	perceptual
3	transformational/ inductive	transformational/ inductive	perceptual	transformational	inductive/ authoritarian	inductive
4	perceptual	axsiomatik	no proof	symbolic	symbolic	perceptual
5	transformational	inductive	transformational	perceptual	inductive	inductive
6	transformational	transformational	transformational/ inductive	authoritarian	inductive	perceptual

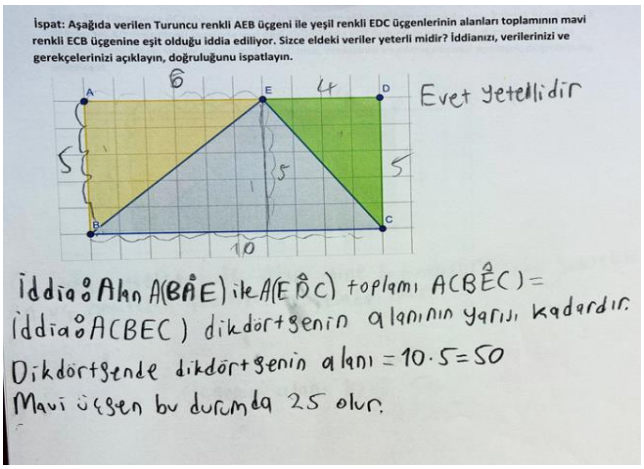
Students produced proof schemes at all three levels; however, the frequency and type of proof schemes differed across achievement groups. Students in the high-achieving group (S1 and S3) were found to produce higher-level proof schemes more frequently. Furthermore, these students generated multiple proof schemes for a single proof task. In contrast, students in the low and moderate-achieving groups predominantly produced empirical proof schemes. Within the moderate-achieving group (S5 and S6), students most frequently produced moderate-level proof schemes, while also generating some high- and low-level schemes. Students in the low-achieving group (S9 and S11) most frequently produced moderate-level proof schemes, but they also generated low-level schemes. Sample student tasks in the process, their proofs, and explanations are presented in Table 7.

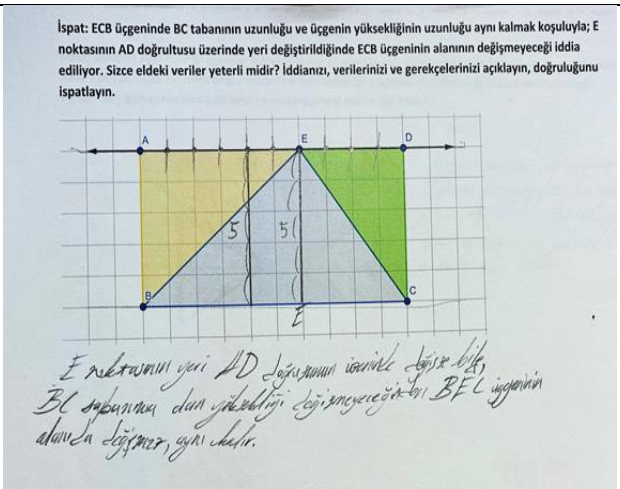
Table 7: *The students' proof schemes*

Students answers	Students proof schemes
Task: It is claimed that the sum of the areas of △ AEB and △ EDC is equal to the area of △ ECB. Do you think the data is sufficient? Explain your claim, ground, and warrant, and prove the validity. R: Which proof method did you use? S1: I proved it using algebraic expressions and supported it with an example.	

R: Do you think we can generalize this method to all examples? S1: I think we can, because I showed an algebraic proof that would be valid for any number.	In the first proof task, S1 used algebraic proof and paragraph proof methods. The student's reasoning consisted of coherent, logically connected steps, demonstrating a transformational proof scheme. However, this reasoning remains one level below the axiomatic proof scheme, as the student did not sufficiently support the argument with formal definitions or axioms. S1's statement indicates that the student perceives the algebraic proof as generalizable. By stating that the proof would be valid "for any number," the student demonstrates an understanding that algebraic reasoning can establish a general mathematical relationship rather than verifying the claim only for specific examples.
Task: It is claimed that two equilateral triangles with the same area will not always be the same lengths and heights. Do you think this claim is true? Prove the claim by making your own claim. Explain your grounds and reasoning, and prove it mathematically. R: How do you know this proof is correct? S3: I used different proof methods and verified the result. R: Could you explain it more clearly? S3: I drew figures and used ratios and algebraic expressions. R: Why did you choose these methods? S3: This claim was false, so I tried to show that it wasn't true.	 S3 demonstrated an axiomatic proof scheme, one of the analytical proof schemes for the fourth task. The student used mathematical notation correctly and appropriately, and also made an effort to continue the proof. The student's reasoning and drawings were related. S3 attempted to support the reasoning through multiple representations and strategies. Furthermore, S3 reported choosing these methods to demonstrate that the given claim was false. This response suggests that the student engaged in refutational reasoning, aiming to disprove the claim by providing counter-supporting arguments. The use

	of various methods also reflects an effort to strengthen the justification of the conclusion.
Task: When the length the same and width is doubled, explain the change in area of a parallelogram. R: Can we generalize this to all parallelograms in a similar way? S5: Yes, the area increases by the same factor as the side increases. R: Are there any special conditions that prevent generalization? S5: What do you mean? R: In your generalization, the height is 2 units, isn't that a limitation? S5: Right, taking the height as 2 units is a limitation; there are parallelograms with different heights. R: How would you correct this? S5: Since the height can change, I would use letters, that is, I would use variables. R: Does it change the validity of claim? S5: The claim doesn't change, but it will always be valid.	 The proof demonstrated by the S5 for the sixth proof task is an example of a transformational proof scheme, one of the analytic proof schemes. S5 initially attempted to generalize the result using a specific numerical value for height. However, after being prompted, the student recognized this limitation and suggested using variables to represent changing heights. This indicates an awareness of the need for symbolic generalization, suggesting a transition from example-based reasoning toward a more general proof structure. The researcher highlighted that assuming a fixed height of 2 units could restrict the generality of the argument. S5 immediately acknowledged this constraint, noting that parallelograms can have different heights. This response also demonstrates that the student identified a flaw in their reasoning, showing metacognitive awareness and the ability to evaluate the assumptions underlying their generalization critically.
Task: Let ABC be a right angled triangle. Write a proof that shows how the area of triangle ABC can be calculated using the area of a rectangle with the same base and height. Write your claim and your grounds. Explain your reasoning and prove it mathematically. R: Do you think this is a proof? S6: Yes	

R: Can you explain why you think it is a proof? S6: I tried to prove the claim with mathematical expressions. R: What mathematical expressions did you use? S6: Square, diagonal, algebraic expressions, etc. R: Is it possible to demonstrate the area of a right triangle using a polygon other than a square? S6: It can also be found with a rectangle.	For the third proof task, S6 was demonstrate a transformational proof scheme. The student began the proof by presenting his claim and continued with definitions and axioms. The student's reasoning is consistent with his claim and drawings. The student used variables rather than specific examples. S6 stated that the solution constituted a proof and explained that the claim was justified using mathematical expressions. When asked to elaborate, the student mentioned using geometric elements such as squares, diagonals, and algebraic expressions during the reasoning process. This indicates that S6 attempted to support the claim through a combination of geometric representation and algebraic reasoning.
Task: It is claimed that the sum of the areas of △ AEB and △ EDC is equal to the area of △ ECB. Do you think the data is sufficient? Explain your claim, ground, and warrant, and prove the validity. R: Is this proof complete? S9: Yes, I wrote my claim, used the properties related to rectangles and triangles, and demonstrated it with an example. R: Do you think this is sufficient? Did you clearly state that the sum of the areas of the yellow and green triangles is equal to the blue? S9: No, I didn't, but the reader can understand it.	 S9 demonstrates an example of an empirical proof scheme. Furthermore, his statement that the proof is understandable also points to a perceptual proof scheme. S9 stated that the solution constituted a proof, noting that the claim was written, relevant properties of rectangles and triangles were applied, and the reasoning was demonstrated with an example. However, when prompted, the student acknowledged that the equality of the areas of the yellow and green triangles to the blue triangle was not explicitly stated, although they believed it could be inferred by the reader.
Task: It is claimed that if the vertex of a triangle is along a line parallel to its base, the area of △ ECB will not change. Do you think the available data are sufficient? Explain your claim, your data, and	

your justification, and prove its validity. R: Is this proof complete? S11: Yes R: Is it clear enough? S11: I provided a more basic level explanation. R: Is it possible to prove it differently? S11: Some additional information can be provided.	 S11 demonstrates an example of a perceptual proof scheme, one of the experimental proof schemes. The student did not clearly explain his hypothesis, leaving gaps for inference. Moreover, the information presented by S11 is accurate.

S1 used algebraic proof and paragraph proof methods in the first proof task. S1's reasoning consists of related, correct mental steps, and it is an example of a transformational proof scheme. The student is one level below the axiomatic proof scheme because he does not receive sufficient support from the definitions or axioms.

S3 demonstrated an axiomatic proof scheme, one of the analytical proof schemes for the fourth task. The student used mathematical notation correctly and appropriately, and also made an effort to continue the proof. The student's reasoning and drawings were related.

The proof demonstrated by the S5 for the sixth proof task is an example of a transformational proof scheme, one of the analytic proof schemes. To obtain the area formula for a parallelogram, the student added congruent right triangles to the sides of the parallelogram.

For the third proof task, S6 demonstrated a transformational proof scheme. The student began the proof by presenting his claim and continued with definitions and axioms. The student's reasoning is consistent with his claim and drawings. The student used variables rather than specific examples.

S9 demonstrates an example of an inductive proof scheme, which is one of the empirical proof schemes. Furthermore, his statement that the proof is understandable also points to a perceptual proof scheme.

S11 demonstrates an example of a perceptual proof scheme, one of the experimental proof schemes. The student did not clearly explain his hypothesis, leaving gaps for inference. Moreover, the information presented by S11 is accurate. These findings are summarized in Table 7.

Discussion and Conclusion

In this study, students' argumentation quality was classified as either "acceptable" or "good argumentation quality". Yackel (2001) noted in her research on argumentation that students are expected to reflect on given proof tasks and provide appropriate justifications and supporting evidence during group work. In most tasks, students clearly defined the claim, data, warrant, backing, and qualifiers, and established appropriate connections among these components. Inglis et al. (2007) argue that Toulmin's framework is effective in facilitating the construction of higher-level arguments in mathematics. Similarly, Cervantes-Barraza et al. (2020) stated that argumentation-based instruction grounded in the Toulmin model is effective at the primary school level for developing mathematical proofs through collaboration. Osborne et al. (2004) observed that students' argumentation quality improved over time in courses designed using Toulmin's argumentation framework. In this study, although students adequately defined other elements of argumentation, they had difficulty formulating rebuttals or identifying limiting cases.

In the second activity, students in the first group did not question the validity of the argument but instead identified a potential limiting case, aligning with Toulmin's definition of a rebuttal. This example aligns with Toulmin's view that while a mathematical proof cannot be refuted, limiting cases may be revealed by altering the problem context to some extent (Aberdein, 2005). According to Kuhn (1993), generating rebuttals to mathematical proofs is a complex cognitive process for children. Because generating a counter-claim requires both content knowledge and higher-order thinking skills, such as evaluation and critical analysis, children often struggle to produce rebuttals (Andrews, 2015; Haro et al., 2020; Kuhn, 2010; Kuhn & Udel, 2003; Walton, 2009).

In this study, students were able to produce external, experimental, or analytical proof schemes. Students who developed analytical proof schemes also employed empirical proof schemes as a secondary means of supporting their arguments. Furthermore, these students were found to construct more organized proof schemes. They defined concepts and axioms and initiated proofs by clearly stating their claims. Students' organized work facilitates the development of analytical proof schemes to analyze and understand the connections between statements and arguments within the proofs (Mariotti et al., 2018; Stylianou et al., 2006). This research has demonstrated that sixth-grade students can produce high-level proof schemes. Therefore, it is important to provide opportunities for proof production in mathematics classes, even at an early age. This study found that students with higher academic achievement produced more advanced forms of proof. One of these students produced four distinct proof schemes.

Lee (2016) stated that broader content knowledge contributes to the production of higher-level evidence schemes. In this study, particularly high-achieving students produced multiple evidence schemes within a single task. This may suggest that students continue to evaluate the proof task critically. Proof plays a fundamental role in advancing

mathematical understanding within the discipline of mathematics (Pedemonte & Balacheff, 2016; Stylianides & Stylianides, 2016). It is well established that well-structured, self-explanatory examples with intuitive solutions can enhance the reasoning and proof skills of middle and low-achieving students in particular (Reiss et al., 2008).

Three main themes emerged from the analysis of standardized open-ended interviews with students regarding APBMI: its perceived usefulness, effectiveness, and collaborative learning. Students with high, medium, and low academic achievement showed slight differences in their responses regarding usefulness thema. The reason students are examined based on academic achievement in this study is the perspective that students with varying levels of academic achievement benefit from instruction in differing degrees (Quinn, 2020).

Analysis of student responses revealed eight positive feedback codes and four negative feedback codes on the usefulness theme. Student responses demonstrated that positive decisions regarding the usefulness theme were more prevalent. The most frequently mentioned positive code regarding the usefulness of the APBMI was “Creating a more engaging and fun learning environment”. This result is notable, as previous research suggests that applying these approaches with younger students may not yield significant effects (Özdemir, 2022). Students with high and medium academic achievement expressed a total of 8 negative opinions on the theme of usefulness, whereas the low-achieving group expressed 7. This suggests that even students who typically perform better academically may encounter challenges or perceive difficulties with argumentation and proof-based activities.

The findings align with the literature emphasizing the pedagogical value of structured discussions in mathematics. Mueller and Yankelewitz (2014) asserted that discussions facilitate rich mathematical dialogue and uncover misconceptions, thereby serving as a valuable practice in the students’ learning process. Argumentation and proof-based practices are pedagogically advantageous, as they foster a rich learning process characterized by extended reflection, opportunities for self-monitoring, critical evaluation of peers’ ideas, and structured feedback (Quinn, 2020). The most frequently expressed negative thought among students was “struggling to provide arguments or evidence”. This finding indicates that constructing evidence is a cognitively challenging activity for students (Blanco et al., 2024; Heinze et al., 2005).

Furthermore, it appears that students struggle not only to reach a conclusion but also to justify the reasoning underlying that conclusion. This difficulty may arise because teaching does not sufficiently emphasize activities such as generating arguments, providing justifications, and making generalizations. In addition, epistemological beliefs that limit students’ perceptions of proof, such as the belief that proof must rely on authority, may also contribute to this difficulty. Another negative thought frequently expressed among students was "Inconsistent viewpoints." Students’ difficulties may stem from limited opportunities to reflect on and critically evaluate their own arguments. In

many classrooms, students are encouraged to get correct answers rather than evaluate the logic of their reasoning. Consequently, students may have limited opportunities to analyze their arguments and identify inconsistencies. Providing students with more opportunities to discuss, question, and revise their arguments can help them develop more consistent viewpoints(Ricco, 2025).

Within the theme of effectiveness, ten distinct positive opinion codes and one negative opinion code emerged. The most frequently mentioned positive code regarding the effectiveness of the APBMI was "Ensuring better learning" and "Establishing the relationship between argument and proof". Students with high and medium academic achievement did not register any negative opinions concerning the theme of effectiveness, whereas the low-achieving group articulated three such opinions. The student perspective that "proof is necessary and important in mathematics" has likewise become a widely accepted discourse within the field of mathematics (Heinze & Reiss, 2007). Szendreı-Radnaı and Török (2007) asserted that proof holds significant importance as it provides systematic methods for mathematical reasoning, exemplifies the logical structure of mathematics, imparts fundamental mathematical concepts, elucidates the meanings and interrelations of these concepts, and fosters both the social and cognitive development of the individual.

These findings suggest that teaching is perceived as an eclectic process that integrates an understanding of the nature of argumentation and proof. Students can organize the arguments they produce into a logical chain and use it as a support mechanism in constructing proofs (Pedemonte, 2007). Students' awareness of the relationship between these two structures indicates that they are beginning to grasp the structural characteristics of mathematical reasoning. This suggests that students understand proof as a structure built upon explanation, justification, and reasoning. The most frequently expressed negative thought among students was their "inability to reflect on different subjects." This finding suggests that students may have perceived the APBMI process as a learning experience specific to a particular subject. This situation may be related to problems with transferring knowledge.

Within the collaborative learning theme, ten positive opinion codes and three negative opinion codes were identified. One of the most frequently repeated phrases by the students was "Critically interpreting the adequacy of evidence presented by peers." Conversely, the high- and medium- achieving groups expressed seven negative opinions regarding collaborative learning, whereas the low-achieving group expressed only two. These findings may account for the significant differences observed in academic achievement. Students who perceived APBMI as effective and useful were predominantly those in higher academic achievement groups. Argumentation-based teaching enables students to compare, explain, justify, discuss, and validate their ideas by presenting them to the whole class (Brown, 2017). Uygun and Akyüz (2019) stated that collective argumentation reveals diverse ways of thinking and questioning. Class discussions constitute a pedagogical approach that enables students to understand and critically

evaluate their peers' cognitive styles while simultaneously facilitating the development of their own ideas (Krummheur, 2015; Stein, 2001).

In this study, the APBT process emerged as a learning environment that directly influenced students' proof-making strategies. During argumentation activities, students explained, justified, and provided evidence for their ideas. Analyses indicated that students used the argumentation process both as an exploratory phase and to produce different proof schemes. Differences in students' proof preferences were observed according to their academic achievement levels. Students in the high-achievement group were able to develop multiple proof schemes for a single task and to employ both analytical and experimental approaches. This indicates that students engaged in the processes of justifying their ideas and generating evidence more flexibly and effectively during argumentation activities. This finding is consistent with the results of Housman and Porter (2003). Students in the medium achievement group produced a limited number of evidence schemes, partially applied analytical schemes, and relied more frequently on experimental schemes.

Additionally, this perception may stem from the prior emphasis on numerous problem-solving examples in earlier mathematics instruction. Students may develop the belief that solving more examples leads to greater success and brings them closer to the correct answer. This suggests that being at a medium achievement level supports some evidence-generating processes while constraining the use of schemes that require analytical thinking and abstraction.

Students in the low-achievement group had difficulty generating both analytical and experimental schemes, often relying on a single scheme to complete the proof tasks. This limitation may be attributed to insufficient content knowledge (Lee, 2016). Furthermore, the students' limited argumentation skills may have contributed, as they did not consistently utilize claims, data, and justifications in the proof tasks. Interviews with students revealed that those in the low achievement group require greater support from teachers or peers to construct their proofs. This finding indicates that low-achieving students require extra support in constructing proofs, emphasizing the need for carefully designed instructional strategies. To address this need, various types of scaffolding can be incorporated into the instructional design to support low-achieving students in developing higher-level proof schemes. Previous research has indicated that supportive structures help students better understand the aims of argumentation and its relationship to proof (Jin & Kim, 2021). In this context, scaffolding can provide resources that support students' current argumentation structures as they develop toward more advanced levels.

Recommendations

In light of this study's findings, a number of recommendations are put forward to inform future research and practice. The sample for this study consisted of sixth-grade students; future studies could include students from different grade levels to accommodate a larger

sample. This study demonstrated that students can develop high-level proof schemes at an early age. In this context, it is recommended that middle school students be actively engaged in proof-related tasks within mathematics instruction. To support this process, such tasks should be accompanied by appropriate scaffolding to promote systematic and organized student work. This study examined students' views on APBMI through semi-structured interviews. Future studies could examine changes in students' attitudes toward teaching and mathematics over time.

Ethical Approval

We wish to confirm that this manuscript has not been published, nor is it under consideration for publication elsewhere. Furthermore, we declare that there are no conflicts of interest associated with this work. Moreover, all necessary ethical permissions for the research were obtained from the Erciyes University Social and Human Sciences Ethics Committee, with approval date October 31, 2023, and issue number 440. The study was conducted with the official approval of the Kayseri Governorship and in accordance with ethical research standards, including obtaining informed consent from all participants. This research is based on the related sections of the Pınar Ormans' doctoral dissertation called “Argumentation and Proof-Based Mathematics Instructional Design” at the Erciyes University, Graduate School of Educational Sciences, under the direction of Prof. Dr. Mustafa Çelebi.

Author Contributions

Pınar Orman: Conceptualisation, Theoretical Background, Methodology, Data Analysis, Data Visualization, Writing, Review and Editing, Funding Acquisition

Mustafa Çelebi: Conceptualisation, Theoretical Background, Methodology, Data Collection, Writing, Review and Editing, Funding Acquisition

Conflict of Interest

The authors declare no conflict of interest.

Plagiarism Checks

Turnitin

Artificial Intelligence Usage Statement

No artificial intelligence tools were used in this study.

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